UNIVERSITY OF PENNSYLVANIA

CONCERNING THE SUBSETS OF A PLANE CONTINUOUS CURVE

HARRY MERRILL GEHMAN

A THESIS

IN MATHEMATICS

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PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

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CONCERNING THE SUBSETS OF A PLANE CONTINUOUS CURVE.*

By Harry Merrill Gehman.

I. Introduction.

In the present paper we shall discuss plane continuous curves† with reference to certain problems concerning the subsets of a continuous curve. It will be understood that the point sets considered lie in a Euclidean space of two dimensions. In part II of this paper is given a method for the construction of a continuous curve from any bounded, closed, and connected point-set by the addition of a countable infinity of arcs, thus showing that any bounded, closed and connected point-set may be a subset of a continuous curve. In part III are given two conditions which are necessary and sufficient, and one condition which is sufficient but not necessary, that every closed and connected subset of a continuous curve be a continuous curve. In part IV is given a characterization of simple continuous curves by means of one of the conditions introduced in part III.

In conclusion, we wish to express our thanks to Professor John Robert Kline, who first suggested the problems which are discussed in this paper, and who, by his never failing encouragement, has materially assisted in their solution.

II.

In this part we shall consider this problem: given a bounded, closed and connected point-set K, which is not a continuous curve; can we always construct a bounded, connected point-set L, such that $K+L$ is a continuous curve? If so, what conditions must the set L satisfy?

The first question is answered in the affirmative by Theorem II, while Theorem I and its corollary state that L must be a set which cannot be expressed as the sum of a finite number of arcs.

Theorem I. *If K is a bounded, closed and connected point-set, which is not a continuous curve, and if L is any arc, then $K+L$ is not a continuous curve.*

Proof. Since K is not a continuous curve, it contains a continu of condensation W, such that K is not connected im kleinen at any point of W.‡

* Various parts of this paper were presented to the American Mathematical Society, April 19, October 25, and December 30, 1924.

† For definitions and theorems concerning continuous curves, see R. L. Moore: *Report on continuous curves from the viewpoint of analysis situs*, Bull. Amer. Math. Soc., vol. 29 (1923), pp. 289–302. This paper will hereafter be referred to as "Report".

‡ Report: pp. 296–97.

Suppose L is an arc which does not contain every point of W, and let P be a point of W which is not a point of L. Since L is closed, about P we can put a circle C, not containing or enclosing any points of L. Since the only points of $K+L$ which C contains or encloses are points of K only, and since K is by hypothesis not connected im kleinen at P, it follows that $K+L$ is not connected im kleinen at P, and therefore that $K+L$ is not a continuous curve.

Suppose L is an arc which contains every point of W. Since W is closed and connected, W is an arc, $\overset{\frown}{AB}$, which is a sub-arc of L. Let P be a point of $\underset{\smile}{AB}$.* Since $L-\underset{\smile}{AB}$ is closed, we can put about P a circle C, not containing any points of $L-\underset{\smile}{AB}$. C will contain no points of L that are not also points of K, and therefore, as in the preceding paragraph, $K+L$ is not connected im kleinen at P, and is not a continuous curve.

We have shown that no matter how the arc L be constructed, $K+L$ is not a continuous curve. The following corollary can be proved by a repeated application of Theorem I.

COROLLARY. *If K is a bounded, closed and connected point-set, which is not a continuous curve, and if L is any point-set which can be expressed as the sum of a finite number of arcs, then $K+L$ is not a continuous curve.*

Note that the next theorem shows the necessity of assuming in this Corollary that the number of arcs into which L is separated, is finite.

THEOREM II. *If K is a bounded, closed and connected point-set, which is not a continuous curve, there exists a bounded connected point-set L consisting of a countable infinity of arcs, such that $K+L$ is a continuous curve.*

PROOF. Let us construct in the plane of K a pair of coördinate axes, and let us select our unit of length so that the diameter of K is greater than 2. Since K is bounded, there is some positive integer n, such that every point of K is at a distance less than n from the origin.

Let L_1 be the sum of the straight line intervals from $(-n, i)$ to (n, i), $(i = 0, \pm 1, \pm 2, \cdots, \pm n)$, and the straight line intervals from (j, n) to $(j, -n)$, $(j = 0, \pm 1, \pm 2, \cdots, \pm n)$.

Let W denote the set of all points of K at which K is not connected im kleinen, plus all limit points of such points. In each square of unit side formed by intervals of L_1, that contains or encloses a point of W, let us join the mid points of opposite sides by segments of lines parallel to the X and Y axes. Let L_2 be the sum of the segments thus added.

In general, let L_i $(i = 3, 4, 5, \cdots)$, be the set of segments added to $L_1+L_2+\cdots+L_{i-1}$, by joining mid points of opposite sides of each square

* If $\overset{\frown}{XYZ}$ is an arc, $\underset{\smile}{XYZ} = \overset{\frown}{XYZ} - X - Z$.

of side $1/2^{i-2}$ formed by intervals of $L_1 + L_2 + \cdots + L_{i-1}$, that contains or encloses a point of W.

Let $L = L_1 + L_2 + \cdots + L_i + \cdots$. Since each L_i consists of a finite number of arcs, L will consist of a countable infinity of arcs. The set L is bounded and connected. L is not necessarily closed, but every limit point of L which is not in L is a point of W. Therefore $K + L$ is bounded, closed and connected.

The set L (and therefore $K + L$) is connected im kleinen at all points, which are not points of K. The set K (and therefore $K + L$) is connected im kleinen at all points, save points of W. It remains to be shown that $K + L$ is connected im kleinen at every point of W.

Let P be a point of W. Let us put about P as center a circle C, of arbitrary radius ε. There exists an integer j such that $1/2^j < \varepsilon$. By our method of formation of L, P is enclosed by a square formed of arcs of $L_1 + L_2 + \cdots + L_{j+3}$, the diameter of which is $\sqrt{2}/2^{j+1}$ or less and therefore less than $1/2^j$. The points of $K + L$, lying within and on this square form a connected set N containing P and lying within the circle C. Hence, if we construct a circle C' having P as center and lying within the square, any point of $K + L$ within the circle C' is joined to P by the connected set N which lies within C. Therefore $K + L$ is connected im kleinen at P, where P is any point of W.

Therefore $K + L$ is a continuous curve, since it is connected im kleinen at every point. This completes the proof of Theorem II.

Another method for the construction of a set L, which is such that $K + L$ is a continuous curve, would be to let L be all points (x, y), such that $|x| \leqq n$, and $|y| \leqq n$. In this case, however, L would consist of an uncountable infinity of arcs, and the continuous curve $K + L$ would be everywhere dense within the square whose vertices are $(\pm n, \pm n)$.

The method outlined in Theorem II is such that $L + W$ is nowhere dense. For suppose the interior of some circle C contained only points of $L + W$. If D, the interior of C, consisted entirely of points of W, then K would be connected im kleinen at all these points, contrary to the definition of W. Some points of D are therefore not points of W. Let P be a point of D which is not a point of W, and therefore is a point of L. Let us construct about P a circle C' within C, and enclosing no points of W. As in the previous discussion, there exists an integer $j > 2$, such that C' encloses a square of side $1/2^{j-2}$ or less, formed from lines of $L_1 + L_2 + \cdots + L_j$, which encloses P. The square encloses no points of W, and therefore the supposition that it encloses a point P of L, contradicts our definition of the construction of L.

We have thus proved that, if L is constructed as in Theorem II, it

follows that, if K is nowhere dense, $K+L$ is nowhere dense, while, if K is dense in any parts, $K+L$ is dense only in those parts in which K is dense.

We can include Theorem I, its Corollary, Theorem II, and this additional property of $K+L$ in a new theorem:

THEOREM III. *If K is any bounded, closed and connected point-set which is not a continuous curve, a point-set L consisting of a countable set of arcs, necessarily infinite in number, can be so constructed that $K+L$ is a continuous curve which is dense only in those parts in which K is dense.*

III.

It is well known that a continuous curve may have the property that each of its closed and connected subsets is also a continuous curve, while another continuous curve may contain closed and connected subsets which are not continuous curves. In the first class is any continuous curve which contains no simple closed curve;* in the second class is any continuous curve which is dense in any part. In this section we obtain necessary and sufficient conditions that a continuous curve be in the first class, i. e., that every closed and connected subset be a continuous curve.

We shall first introduce and discuss several definitions.

Definition. If M and W are point-sets, then M is said to be *connected im kleinen relative to* W at a point P, if for every positive number ϵ, there exists a positive number δ, such that, if X and Y are any two points of M at a distance less than δ from P, then they lie in a connected subset of M, which is such that either (1) every point is at a distance less than ϵ from P, or (2) every point is at a distance less than ϵ from a point of W.

Definition. If M and W are point-sets, then M is said to be *uniformly connected im kleinen relative to* W, if for every positive number ϵ, there exists a positive number δ, such that if X and Y are any two points of M at a distance apart less than δ, then X and Y lie in a connected subset of M which is such that either (1) it lies within a circle of radius ϵ, or (2) every point is at a distance less than ϵ from a point of W.†

If M is a closed and bounded set and is connected im kleinen relative to W, then it is uniformly connected im kleinen relative to W, but this is not necessarily true in case M is open or unbounded.

The property of being *relatively uniformly connected im kleinen* is weaker

* S. Mazurkiewicz, *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 2 (1921), p. 123.

† This notion is somewhat akin to that of relatively uniform convergence of functions, due to E. H. Moore. See *The New Haven Mathematical Colloquium.*

than that of being *uniformly connected im kleinen*, and the property of being *relatively connected im kleinen* is weaker than that of being *connected im kleinen*. The properties of being *relatively uniformly connected im kleinen* and of being *connected im kleinen* are independent of each other; a set may have either property without having the other. For example, if W is a point on a simple closed curve J and M is $J - W$, then M is connected im kleinen but is not uniformly connected im kleinen relative to W. On the other hand, if M is the curve $y = \sin\frac{1}{x}$, for $0 < x \leqq 1$ plus the interval of the Y-axis between $y = 1$ and $y = -1$, and if W is the portion of the Y-axis included in M, then M is not connected im kleinen but it is uniformly connected im kleinen relative to W.

For the special case where W is closed and has no points in common with M, the property of being *relatively uniformly connected im kleinen* is stronger than that of being *connected im kleinen*. We have given above an example in which M is connected im kleinen but not uniformly connected im kleinen relative to W. If, however, a set M is uniformly connected im kleinen relative to W, where W is closed and has no points in common with M, then M is connected im kleinen. For if it were not, it would fail to be connected im kleinen at a point P. Let ϵ be less than one-third the distance from P to any point of W. For any number δ, there are points in M at a distance from P less than δ that cannot be joined to P either by a connected set in M which lies within a circle of radius ϵ or by a connected set which is such that every point is at a distance less than ϵ from a point of W. The set M is therefore not connected im kleinen relative to W, either uniformly or non-uniformly, contrary to hypothesis.

Therefore in case W is closed and has no points in common with M, the property of being *relatively uniformly connected im kleinen* lies between the properties of being *connected im kleinen* and of being *uniformly connected im kleinen*. The property S defined by R. L. Moore* is also a property which lies between these two properties. The following examples show, however, that, if W is closed and has no points in common with M, the property of being *relatively uniformly connected im kleinen* and the property S are independent of each other. For if W is a point on a simple closed curve J, and M is $J - W$, then M has property S, but is not uniformly connected im kleinen relative to W. If M is the curve $y = \sin\frac{1}{x}$, for $0 < x \leqq 1$, and W is $x = 0$ for $-1 \leqq y \leqq 1$, then M is uniformly connected im kleinen relative to W but does not have property S.

* Report, p. 296. "*A point set M has property S* if, for every positive number ϵ, M is the sum of a finite number of connected subsets each of diameter less than ϵ."

Definition. *An R—set* is a closed and connected point set N, containing a continu of condensation W, such that $N-W$ is uniformly connected im kleinen relative to W.

THEOREM IV. *If M is a continuous curve, a necessary and sufficient condition that every closed and connected subset of M be a continuous curve, is that M contain no R-sets.*

Before proceeding with the proof of Theorem IV, we shall prove two lemmas which are needed in the proof.

LEMMA *A.* *If A is a connected set, and if B is a connected subset of A, such that $A-B$ is disconnected, and if $A-B=H_1+H_2$ is any method of expressing $A-B$ as the sum of two mutually separated* sets, then $B+H_1$ and $B+H_2$ are connected.*

PROOF. Suppose $B+H_1$ were disconnected. Then $B+H_1$ can be expressed as the sum of two mutually separated sets T_1+T_2. Since B is connected, B lies entirely in one of these sets, say T_1. In that case, T_2 is a subset of H_1. Now the set $A=(B+H_1)+H_2=T_1+T_2+H_2=(T_1+H_2)+T_2$. But T_1 and T_2 are mutually separated, and so are H_2 and T_2, because H_1 and H_2 are mutually separated and T_2 is a subset of H_1. Therefore A it expressed as the sum of the two mutually separated sets: (T_1+H_2) and T_2, and A is therefore disconnected, contrary to hypothesis.

Therefore $B+H_1$ is connected. A similar proof shows that $B+H_2$ is connected.

LEMMA *B.* *Any maximal† connected subset of a continuous curve contained in a simple closed curve plus its interior, is also a continuous curve.*

PROOF. Let M be a continuous curve; J a simple closed curve; I the interior of J; K a maximal connected subset of M lying in $J+I$. We shall prove that K is a continuous curve.

The set K is closed and connected. We shall next show that K is connected im kleinen at every point. Let P be a point of K interior to J. Since the continuous curve M is connected im kleinen at P, if we construct any circle J_1 interior to J and having P as center, there can be constructed another circle J_2, interior to J_1 and having P as center, such that every point of M interior to J_2 can be joined to P by an arc in M interior to J_1. As K is a maximal connected set all such points and all points of such arcs lie in K, and therefore we can say that corresponding to the circle J_1 there exists the circle J_2 such that every point of K interior to J_2 can be joined to P by an arc in K interior to J_1.

* Two sets are said to be *mutually separated* if neither set contains a point or a limit point of the other set.

† A set A is said to be a *maximal connected subset* of a set B, if A is a connected subset of B which is not a proper subset of any other connected subset of B.

In case a circle J_3 is constructed about P as center but not interior to J, we can construct a circle J_1 about P as center which is interior to both J_3 and J. As above, a circle J_2 exists such that every point of K interior to J_2 can be joined to P by an arc in K interior to J_1 and therefore interior to J_3. Therefore K is connected im kleinen at P, where P is any point interior to J.

Let Q be a point of K on J. If K is not connected im kleinen at Q, it fails to be connected im kleinen at any point of a continu of condensation L. Since K is connected im kleinen at all points interior to J, it follows that L is a subset of J, and is therefore either an arc or the simple closed curve J. Let Q_1 be any point of L, but not an end point in case L is an arc. As Q_1 is a point at which K fails to be connected im kleinen, there is a circle J_4 about Q_1 as center such that no circle J_5 can be constructed within J_4 and with center at Q_1 and such that every point of K within J_5 can be joined to Q_1 by an arc in K lying within J_4. We shall obtain the desired contradiction by showing that such a circle J_5 can be constructed.

If $Q_2 Q_1 Q_3$ be an arc in L and interior to J_4, we can construct an arc joining Q_2 and Q_3, interior to J_4, and interior to J save for its end points. We can also construct an arc joining Q_2 and Q_3, interior to J_4, and exterior to J save for its end points. These two arcs form a simple closed curve J_6, which contains Q_2 and Q_3 and encloses $Q_2 Q_1 Q_3$, and which is interior to J_4. Since the continuous curve M is connected im kleinen at Q_1, corresponding to J_6 is a circle J_5 interior to J_6 and having Q_1 as center such that every point of M interior to J_5 can be joined to Q_1 by an arc in M interior to J_6. If any point of M interior to J and to J_5 is joined to Q_1 by an arc interior to J_6, and having points exterior to J, this arc has a first point in common with $Q_2 Q_1 Q_3$, since $Q_2 Q_1 Q_3$ is the subset of J which separates the interior of J_6 into points interior to J and points exterior to J. Therefore any point of M interior to J and to J_5 can be joined to Q_1 by an arc lying in $J+I$, and is therefore a point of K, as are all points on these arcs. Therefore all points of K interior to this circle J_5 can be joined to Q_1 by an arc in K interior to J_6 and therefore interior to J_4. Thus our supposition that K is not connected im kleinen at Q has lead to a contradiction.

Therefore K is connected im kleinen at every point, and is therefore a continuous curve.

Proof of Theorem IV. We shall first prove that the given condition is necessary by showing that an R-set either is not a continuous curve, or contains a closed and connected subset which is not a continuous curve. That sets of the first type exist may be seen by the aid of the following

example in which the set N consists of W, which is the interval of the Y-axis between $y = 1$ and $y = -1$, and $N - W$, which is the curve, $y = \sin\frac{1}{x}$, for $0 < x \leqq 1$.

Let us suppose then that N is an R-set which is a continuous curve. Since $N - W$ is uniformly connected im kleinen relative to W, given $\varepsilon_1 > 0$, there exists a number $\delta_1 > 0$, such that if two points of $N - W$ are at a distance apart less than δ_1, they can be joined by a connected subset of $N - W$, which either lies entirely within a circle of radius ε_1, or is such that every point is at a distance less than ε_1 from a point of W. They can therefore be joined by an arc in $N - W$, satisfying one of the two conditions.*

If we put about each point of W as center a circle of radius $\delta_1/4$, by the Heine-Borel Theorem we can select from this infinite set a finite subset of circles, $C_1, C_2, \cdots, C_n$, covering W. Let P_i $(i = 1, 2, \cdots, n)$ be a point of $N - W$ lying within C_i. Since W is connected, each circle of $C_1 + C_2 + \cdots + C_n$ will intersect at least one other circle. We will arrange the set of circles in some order, $C_{n_1}, C_{n_2}, \cdots, C_{n_k}$, such that (1) each circle occurs at least once, and (2) each circle intersects both the preceding and the following circles in this arrangement. If C_i and C_j intersect $(i, j = 1, 2, \cdots, n)$, P_i and P_j are at a distance apart less than δ_1, and can therefore be joined by an arc in $N - W$, every point of which is at a distance less than either $(2\varepsilon_1 + \frac{1}{4}\delta_1)$ or ε_1, and therefore less than $3\varepsilon_1$, from a point of W.

If we construct the arcs, $P_{n_1} P_{n_2}$, $P_{n_2} P_{n_3}$, $\cdots$, $P_{n_{k-1}} P_{n_k}$, their sum is a continuous curve K_1, having no points in common with W, and such that every point of K_1 is at a distance less than $3\varepsilon_1$ from some point of W, and every point of W is at a distance less than $\delta_1/2$ (and therefore less than $\varepsilon_1/2$) from some point of K_1.

If $3\varepsilon_2$ is less than the distance from any point of W to any point of K_1, we will construct a continuous curve K_2 corresponding to ε_2 in the same way that K_1 corresponds to ε_1.

By this process we obtain an infinite sequence of continuous curves, $K_1, K_2, K_3, \cdots$, each one having no points in common with any other one or with W, and such that every point of K_i $(i = 1, 2, 3, \cdots)$ is at a distance less than $3\varepsilon_i$ from some point of W, and every point of W is at a distance less than $\varepsilon_i/2$ from some point of K_i. For $i \geqq 2$, $\varepsilon_i < \varepsilon_{i-1}/6$. From these considerations, it follows that the only limit points of $(K_1 + K_2 + K_3 + \cdots)$ are in W, and every point of W is a limit point of $(K_1 + K_2 + K_3 + \cdots)$.

* Report, p. 294. See also: R. L. Moore, *Concerning continuous curves in the plane*, Math. Zeit., vol. 15 (1922), p. 255.

Let P be a point of W. Let α be a positive number such that there are points of W at a distance greater than α from P. Since N is connected im kleinen at P, given $\alpha > 0$, there exists a number $\beta > 0$ such that any point of N at a distance less than β from P can be joined to P by an arc in N of diameter less than α. Only a finite number of the sets, $K_1, K_2, K_3, \cdots$, have no points closer to P than β, and therefore there exists an integer m such that, for $i \geqq m$, K_i has points at a distance less than β from P. Let Q_i $(i \geqq m)$ be a point of K_i at a distance less than β from P. Let us join each of the points, $Q_m, Q_{m+1}, Q_{m+2}, \cdots$, to P by an arc in N lying within a circle of radius α having P as center, which circle we shall denote by C.

We will now show that $K = W + (K_m + K_{m+1} + \cdots) + (\widehat{Q_m P} + \widehat{Q_{m+1} P} + \cdots)$ + limit points is a subset of N which is closed and connected, but which is not a continuous curve because it is not connected im kleinen. The set K is connected, by construction, because it consists of a countable infinity of connected sets [W, and $(K_i + \widehat{Q_i P})$], each of which contains P, forming a connected set which remains connected upon the addition of its limit points. But K is also closed, since we have included in K the limit points of $(\widehat{Q_m P} + \cdots)$, all of which lie within or on the circle C.

Let A be a point of W exterior to C. If ε is less than the distance from A to any point of C, then although there are points of K [i. e., points of $(K_m + K_{m+1} + \cdots)$] arbitrarily close to A, these points can be joined to A only by a connected set in K which lies partly within C, and therefore is of diameter greater than ε.

Therefore K is not connected im kleinen at A, and is therefore not a continuous curve.

Continuing our proof of Theorem IV, we shall next prove that the given condition is sufficient by showing that, if some closed and connected subset K of a continuous curve M is not a continuous curve, then M contains an R-set.

By means of R. L. Moore's characterization of closed and connected sets which are not continuous curves,* it is evident that K must satisfy the following conditions: there exists a square $ABCD$ in the plane, and a countable infinity of closed and connected sets, $W, K_1, K_2, K_3, \cdots$, such that (1) each of these sets is a subset of K, has at least one point on AD and at least one point on BC, has no points on AB or CD, and is a subset of the set H consisting of $ABCD$ and its interior, (2) no two of these sets have a point in common, and no one of them (save possibly W) is a proper subset of any connected point-set which is common to K and H,

* Report, pp. 296–97.

(3) the set W is the sequential limiting set* of the sequence of sets, $K_1, K_2, K_3, \cdots$.

From (1) and (2), if K' is any one of the sets, $W, K_1, K_2, K_3, \cdots$, then $H - K'$ is disconnected into at least two connected parts, of which one contains AB and a second contains CD, and no connected part save these two contains any point of $W + K_1 + K_2 + K_3 + \cdots$. In the case of $H - W$, either H_1, the connected part containing AB, or H_2, the connected part containing CD, contains an infinite number of sets of $K_1, K_2, K_3, \cdots$. Suppose H_1 contains an infinite number.

Let E be a point of W interior to $ABCD$. If we put a circle of radius ϵ about E as center, ϵ being chosen so that the circle is interior to $ABCD$, then, since M is a continuous cnrve, there exists a number δ, such that any point of M within a circle of radius δ about E as center can be joined to E by an arc in M within the circle of radius ϵ. By (3) the circle of radius δ contains a point F of K_a, where K_a is one of the sets, $K_1, K_2, K_3, \cdots$, which lie in H_1. The arc EF in M and within the circle of radius ϵ contains as a subset an arc E_1F_1 such that E_1 is in W, F_1 is in K_a, E_1F_1 has no points in common with either W or K_a. E_1F_1 lies, except for E_1, in H_1.

Let N_1 be the maximal connected subset of M lying in H_1 and containing F_1. Evidently N_1 will contain E_1F_1, the set K_a, and an infinite number of other sets of $K_1, K_2, K_3, \cdots$. Since any limit point of H_1 not contained in H_1 is contained in W, all limit points of N_1 which are not in N_1 are in W. Since N_1 contains an infinite number of the sets: $K_1, K_2, K_3 \cdots$, and since W is their sequential limit, it follows that every point of W is a limit point of N_1. Therefore $N_1 + W$ forms a closed set, which we shall designate by N.

We shall now show that N is an R-set. N is closed and connected, and W is a continu of condensation of N. We shall next prove that $N - W$ is uniformly connected im kleinen relative to W.

Let us select a positive number ϵ. From (3), corresponding to ϵ there is a positive integer n such that, for $i > n$, every point of K_i and every point of that connected part S of $H_1 - K_i$ which has limit points in W

* The point-set M is said to be the *limiting set of the sequence of point-sets* $M_1, M_2, M_3, \cdots$ provided that (a) each point of M is the sequential limit point of an infinite subsequence of some sequence of points, $P_1, P_2, P_3, \cdots$, such that, for every n, P_n belongs to M_n, (b) if $P_1, P_2, P_3, \cdots$ is a sequence of points such that, for every n, P_n belongs to M_n, then M contains the sequential limit point of every subsequence of $P_1, P_2, P_3, \cdots$ that has a sequential limit point. If the further condition is satisfied that every infinite subsequence of the sequence, $M_1, M_2, M_3, \cdots$, has the same limiting set M, then M is said to be the *sequential limiting set* of the sequence $M_1, M_2, M_3, \cdots$.

is at a distance less than ε from some point of W. Let a number i be chosen such that $i > n$ and such that K_i is in N. We shall denote $S + K_i$ by H_3.

The set $N - W$ is connected, because $N - W = N_1$, and N_1 is connected by definition. Therefore any two points X and Y of $N - W$ in H_3 can be joined by a connected set in $N - W$. If this connected set has points in $H_1 - H_3$, the removal of K_i disconnects it into the subsets lying in $H_3 - K_i$ and in $H_1 - H_3$. By Lemma A, the subset in $H_3 - K_i$ and the set K_i form a connected set joining X and Y in H_3. Therefore any two points in H_3 can be joined by a connected set in H_3, and therefore such that every point is at a distance less than ε from a point of W.

In the set $S = H_3 - K_i$ let us construct an arc $C_1 D_1$ such that C_1 is on $\overset{\frown}{BC}$, D_1 is on $\overset{\frown}{AD}$, and $\overset{\frown}{C_1 D_1}$ is interior to $ABCD$. The arc $C_1 D_1$ and the arcs $D_1 A$, $\overset{\frown}{AB}$, and $\overset{\frown}{BC_1}$ of $ABCD$ form a simple closed curve J which contains or encloses all points of K_i, and which contains W in its exterior. Let L be the maximal connected set of M lying in J plus its interior and containing K_i. Since J plus its interior is a subset of H_1, L is a subset of N. By Lemma B, L is a continuous curve.

Since L is a continuous curve, L is uniformly connected im kleinen. Therefore corresponding to our given ε, there exists a positive number δ_1 such that, if the distance between any two points of L is less than δ_1, the two points lie in a connected subset of L which lies in a circle of radius ε.

No point of L save points on $C_1 D_1$ can be a limit point of points of $N - L$. Therefore there is a positive number δ_2 which is less than the distance from any point of K_i to any point of $N - L$. Note also that all points of $N - L$ lie in $S + W = H_3 - K_i + W$ and none lie in $H_1 - H_3$.

We shall now show that, if δ is selected so as to satisfy the inequalities $\delta < \delta_1$, $\delta < \delta_2$, then, if X and Y are any two points of $N - W$ at a distance apart less than δ, they can be joined by a connected subset of $N - W$ which is such that either it lies within a circle of radius ε, or every point is at a distance less than ε from a point of W.

If X and Y are points of $N - W$ in H_3, we have shown that they can be joined by a connected set such that every point is at a distance less than ε from a point of W. If X is a point of $N - W$ in $H_1 - H_3$, it is a point of L and, since $\delta < \delta_2$, Y is also a point of L. Then, since $\delta < \delta_1$, X and Y can be joined by a connected subset of L (and therefore of $N - W$) lying in a circle of radius ε.

Therefore N is an R-set, which completes the proof of Theorem IV.

THEOREM V. *If M is a continuous curve, a necessary and sufficient condition that every closed and connected subset of M be a continuous curve is*

that, given any positive number ε, M contains at most a finite number of mutually-exclusive closed and connected sets of diameter greater than ε.

PROOF. The given condition is sufficient. For if some closed and connected subset N of M is not a continuous curve, then from R. L. Moore's characterization of closed and connected sets which are not continuous curves it follows that N will contain an infinite number of closed and connected sets of diameter greater than some fixed number ε, and no two of the sets will have any points in common.

To show that the condition is necessary, we shall show that, if every closed and connected subset of M is a continuous curve, the supposition that M contains an infinite number of mutually-exclusive closed and connected sets of diameter greater than some fixed number ε leads to a contradiction. Let these sets be $M_1, M_2, \cdots$.

In $M_i\,(i = 1, 2, 3, \cdots)$ let A_i and B_i be two points whose distance apart is more than ε. Since M_i is a continuous curve, by hypothesis, there is an arc $A_i B_i$ in M_i whose diameter is more than ε. Let A be a limit point of the points $A_1, A_2, \cdots$. From the set of points $A_1, A_2, \cdots$, select a subset $A_{n_1}, A_{n_2}, \cdots$, approaching A as a sequential limit point. If B is a limit point of the corresponding set $B_{n_1}, B_{n_2}, \cdots$, then we can select a subset of $B_{n_1}, B_{n_2}, \cdots$, say $B_{j_1}, B_{j_2}, \cdots$, approaching B as a sequential limit point. If we replace A_{j_i} by C_i $(i = 1, 2, \cdots)$ and B_{j_i} by D_i $(i = 1, 2, \cdots)$, we have this state of affairs: an infinite sequence of arcs $C_1 D_1, C_2 D_2, \cdots$ in M, no pair having any points in common, such that $C_1, C_2, \cdots$ approach A as their sequential limit point, the distance from C_i to D_i $(i = 1, 2, \cdots)$ being more than ε.

An arc $C_1 A$ can be passed through all the points of $C_1, C_2, \cdots$, and an arc $D_1 B$ can be passed through $D_1, D_2, \cdots$.* Moreover, these arcs can be drawn so that they have no points in common and so that they have only C_1 and D_1 in common with the arc $C_1 D_1$ of M. Each of the arcs $C_2 D_2, C_3 D_3, \cdots$, of M contains a subarc, $E_2 F_2, E_3 F_3, \cdots$, such that E_i is on $C_1 A$, F_i is on $D_1 B$, and $E_i F_i$ has no other points in common with $C_1 A$ or $D_1 B$.

From the set of arcs $E_2 F_2, E_3 F_3, \cdots$, we can select a subset $E_{n_1} F_{n_1}, E_{n_2} F_{n_2}, \cdots$ such that, if $E_j F_j$ and $E_k F_k$ are any two distinct arcs of the subset, then either $E_j F_j$ is interior (save possibly for its end points) to the simple closed curve $E_k F_k D_1 C_1$, or $E_k F_k$ is interior (save possibly for its end points) to the simple closed curve $E_j F_j D_1 C_1$.

Let E be a limit point of the set $E_{n_1}, E_{n_2}, \cdots$. The point E will be on the arc $C_1 A$. Let $G_1, G_2, \cdots$ be a subset of the set $E_{n_1}, E_{n_2}, \cdots$ which

*R. L. Moore and J. R. Kline, *On the most general plane closed point-set through which it is possible to pass a simple continuous arc*, Ann. of Math., vol. 20 (1919), p. 218.

approaches E as a sequential limit point, and such that G_i lies between G_j and E on the arc $C_1 A$, if $j < i$. Except possibly for a finite number of points, the corresponding points $H_1, H_2, \cdots$ of the set $F_2, F_3, F_4, \cdots$ will occur on $D_1 B$ in the same order as the points $G_1, G_2, \cdots$ occur on $C_1 A$. Suppose this holds true for all points after H_n. If F is a limit point of the set $H_n, H_{n+1}, \cdots$, it cannot lie between any two of them, nor can H_n lie between F and H_{n+1}, and F is therefore a sequential limit point of $H_n, H_{n+1}, \cdots$ and H_i lies between H_j and F on the arc $D_1 B$ if $n \leqq j < i$.

We have now an infinite sequence of arcs $G_n H_n, G_{n+1} H_{n+1}, \cdots$, such that G_i is on $C_1 A$, and H_i is on $D_1 B$, and such that the simple closed curve $G_i H_i H_j G_j$ contains and encloses the arc $G_k H_k$ if $i < k < j$. Let W denote the set of limit points of $G_1 H_1, G_2 H_2, \cdots$; W is closed and contains E and F. If W were not connected, we could enclose V, the largest connected part of W that contains E, by a simple closed curve J not containing any points of W and having points of $W - V$ in its exterior.*

Let P be a point of W exterior to J. A circle J' exterior to J about P encloses points of infinitely many of the arcs $G_n H_n, G_{n+1} H_{n+1}, \cdots$. Since J encloses E, J encloses points of all but a finite number of the arcs $G_n H_n, \cdots$. There are therefore an infinite number of arcs having points interior to J and also points exterior to J (i. e., interior to J'), and therefore having points on J. Some limit point of $G_n H_n, G_{n+1} H_{n+1}, \cdots$ therefore lies on J; that is, some point of W is on J, contrary to our hypothesis on J. Therefore W is connected.

The set W, being a closed and connected subset of M, is a continuous curve. Draw the arc EF in W. If $W - EF \neq 0$, every point of $W - EF$ is enclosed by the simple closed curve $EF H_n G_n$. Note also that no one of the arcs $G_n H_n, G_{n+1} H_{n+1}, \cdots$ has any points in common with W, otherwise our hypotheses on $G_n H_n, G_{n+1} H_{n+1}, \cdots$, are contradicted.

The set M is connected im kleinen at E. Therefore, if we put a circle about E such that F lies in its exterior, there exists a number $k \geqq n$, such that, if $i \geqq k$, an arc $G_i E$ in M lies within the circle.

Let N be $G_k H_k + G_{k+1} H_{k+1} + \cdots$ plus $G_k E + G_{k+1} E + \cdots$ plus W plus limit points of $G_k E + G_{k+1} E + \cdots$ all of which lie within the circle about E. The set N is a closed and connected subset of M, which is therefore a continuous curve. But N is not connected im kleinen at F because, if a circle is put about F having no points in common with the one about E, it will contain points of N arbitrarily close to F, these points being points of $G_k H_k + G_{k+1} H_{k+1} + \cdots$, and, since each of these arcs is distinct and has no points in common with W, points on different arcs can be joined

* L. Zoretti, *Sur les fonctions analytiques uniformes*, Journ. de Math., 6e ser., vol. 1 (1905), p. 10.

in N only by connected sets having points within the circle about E and therefore exterior to the circle about F.

We have thus obtained the desired contradiction and the proof of Theorem V is completed.

THEOREM VI. *If M is a continuous curve, a sufficient (but not necessary) condition that every closed and connected subset of M be a continuous curve is that, given any positive number ε, M can be expressed as the sum of a finite number of closed and connected point-sets each of diameter less than ε, and each pair of sets having at most a finite number of points in common.*

PROOF. Let $M = M_1 + M_2 + \cdots + M_n$, each a closed and connected set of diameter less than ε, and let the points of M common to two or more of these sets be $P_1, P_2, \cdots, P_k$. We shall determine the maximum possible number of mutually exclusive closed and connected sets of diameter greater than ε that are contained in M.

Any such set does not lie entirely in any one of the sets $M_1, M_2, \cdots, M_n$ and, being connected, must contain at least one of the points $P_1, P_2, \cdots, P_k$. No two of the sets of diameter greater than ε can contain the same point, P_i, as the sets are mutually exclusive. Therefore k is the maximum number of such sets.

Therefore, by Theorem V, every closed and connected subset is a continuous curve.

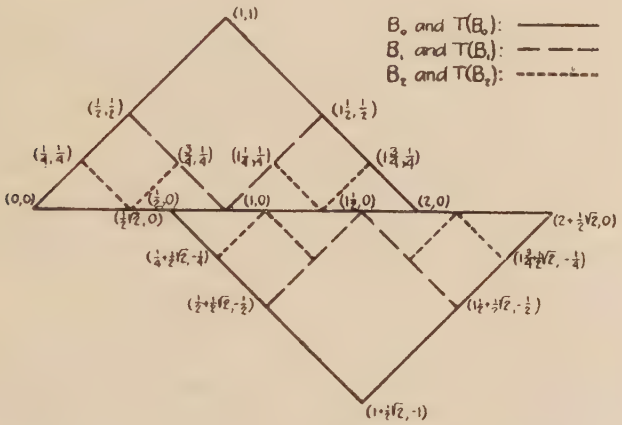
The following example shows that the condition is not necessary. Let B_0 be the straight line intervals from $(0, 0)$ to $(2, 0)$, from $(2, 0)$ to $(1, 1)$, and from $(1, 1)$ to $(0, 0)$. Let B_1 be the straight line intervals from $(1, 0)$ to $(\frac{1}{2}, \frac{1}{2})$; from $(1, 0)$ to $(1\frac{1}{2}, \frac{1}{2})$. Let B_2 be the straight line intervals from $(\frac{1}{2}, 0)$ to $(\frac{1}{4}, \frac{1}{4})$; from $(\frac{1}{2}, 0)$ to $(\frac{3}{4}, \frac{1}{4})$; from $(1\frac{1}{2}, 0)$ to $(1\frac{1}{4}, \frac{1}{4})$; from $(1\frac{1}{2}, 0)$ to $(1\frac{3}{4}, \frac{1}{4})$. In general, the intervals of $B_i\,(i = 1, 2, 3, \cdots)$ form with the intervals of $B_0 + B_1 + \cdots + B_{i-1}$ a set of 2^i isosceles triangles, each of which is such that (1) the base is on the X-axis and is of length $1/2^{i-1}$, (2) the x-coördinates of the end points of the base occur among the numbers $\frac{0}{2^{i-1}}, \frac{1}{2^{i-1}}, \frac{2}{2^{i-1}}, \frac{3}{2^{i-1}}, \cdots, \frac{2^i}{2^{i-1}}$, (3) the sides have the slopes $+1$ and -1, (4) the y-coördinate of the vertex is $1/2^i$. Given B_i, B_{i+1} is the set of 2^{i+1} intervals formed by joining, in each triangle, the mid point of the base to the mid point of each side.

Let $A_1 = B_1 + B_2 + B_3 + \cdots$. Let A_2 be the set (x', y') obtained by subjecting the points (x, y) of A_1 to the transformation $x' = x + \frac{1}{2}\sqrt{2}$, $y' = -y$. Let $M = A_1 + A_2$. (See Figure.)

That every closed and connected subset of M is a continuous curve is best seen by means of Theorem IV. We shall next show that there exists

no finite set of points in M such that M can be expressed as the sum of a finite number of closed and connected sets each of diameter less than 1/2 and each pair having in common only points of this finite set.

Suppose such a finite set of points did exists in M. Call them $P_1, P_2, \cdots, P_k$. On the straight line interval from $(\frac{1}{2}\sqrt{2}, 0)$ to $(2, 0)$, which is common to A_1 and A_2, there occur at least two points of the set, say P_1 and P_2. If the x-coördinate of P_1 is not of the form $a/2^b$, where a and b are integers, P_1 is the sequential limit of a sequence of isosceles triangles in A_1, whose bases contain P_1, each triangle in the sequence containing and enclosing the triangles that follow. Similarly, unless the x-coördinate of P_1 is of the form $a/2^b + \frac{1}{2}\sqrt{2}$, where a and b are integers, P_1 is the sequential limit of a similar sequence of triangles in A_2. Since the



x-coördinate of P_1 cannot be of both these forms, one of the above cases holds for any point P_1. Suppose in this case that the x-coördinate of P_1 is not of the form $a/2^b$.

One of the sets $M_1, M_2, \cdots, M_n$ contains P_1 and an interval of the X-axis having one end at P_1 and the other end between $(0, 0)$ and P_1. Call this set M_1. One of the sets, say M_2, contains P_1 and an interval of the X-axis having one end at P_1 and the other end between P_1 and $(2 + \frac{1}{2}\sqrt{2}, 0)$. Since by hypothesis the x-coördinate of P_1 is not of the form $a/2^b$, there is an infinite sequence of triangles in A_1, one end of whose base lies in M_1, and the other end in M_2. If α is a number less than the distance from P_1 to any of the points $P_2, P_3, \cdots, P_k$, there is a triangle in A_1 of diameter less than α whose base contains P_1 and whose other two sides join a point of M_1 to a point of M_2 by an arc not containing any points of $P_1 + P_2 + \cdots + P_k$. Therefore $M_1 = M_2$.

If the x-coordinate of P_1 is also not of the form $a/2^b + \frac{1}{2}\sqrt{2}$, P_1 is interior to a simple closed curve in M of diameter less than α, and therefore entirely in M_1. In this case none of the sets $M_2, M_3, \cdots, M_n$ contain P_1.

If the x-coordinate of P_1 is of the form $a/2^b + \frac{1}{2}\sqrt{2}$, P_1 is an end point of two straight line intervals of slope $+1$ and -1 in A_2. Denote one of these by $P_1 Q$. Let M_3 denote the set which contains P_1 and an interval of $P_1 Q$ having one end at P_1. There is an infinite set of line intervals in A_2, perpendicular to $P_1 Q$, joining $P_1 Q$ to the X-axis, and approaching P_1 as a sequential limit. As before, there are an infinite number of these at a distance less than α from P_1 and therefore joining a point of M_1 to a point of M_3 by an arc not containing any points of $P_1 + P_2 + \cdots + P_k$. Therefore $M_1 = M_3$. Therefore, if M_1 contains P_1, it contains an interval of each of the straight line intervals in A_2 having an end at P_1. In this case also there is a minimum distance from P_1 to a point of any of the sets $M_2, M_3, \cdots, M_n$, and therefore none of them contain P_1.

Therefore P_1 can be eliminated from our finite set of points and $P_2 + P_3 + \cdots + P_k$ will serve as well. Similarly we can eliminate any other point lying on the interval from $(\frac{1}{2}\sqrt{2}, 0)$ to $(2, 0)$, in which case this interval lies entirely in one of the sets, say M_1. M_1 is then of diameter greater than 1/2, contrary to hypothesis. This shows that the condition is not necessary.

IV.

In part III we have introduced the idea of a point-set being *uniformly connected im kleinen relative to another point-set.* This new idea gives us a characterization of the class of point-sets which R. L. Moore has called simple continuous curves,* namely the simple continuous arc, the simple closed curve, the ray, and the open curve,—the point-sets which are in continuous $(1-1)$ correspondence† respectively with an interval of a straight line, a circle, a half-line, and a line.

THEOREM VII: *If M is a closed and connected plane point-set, such that $M-W$ is uniformly connected im kleinen relative to W, where W is any closed and connected subset of M consisting of more than a single point, then M is a simple continuous arc, a simple closed curve, a ray, or an open curve.*

PROOF. We have shown in part III that if $M-W$ is uniformly connected im kleinen relative to W, where W is closed, then $M-W$ is connected im kleinen. Since W is arbitrary, it follows that M is connected im kleinen

* R. L. Moore, *Concerning simple continuous curves*, Trans. Amer. Math. Soc., vol. 21 (1920), pp. 333–47.

† A correspondence T which sends M into $T(M)$ is said to be *continuous* if, in case the point P of M is a limit point of N, a subset of M, then $T(P)$ is a limit point of $T(N)$.

at every point, and is therefore a continuous curve. There are two cases to be considered.

CASE I. M contains a simple closed curve J. Since by hypothesis $M-J$ is uniformly connected im kleinen relative to J, no point of J is a common limit point of a set of points of M interior to J and a set exterior to J.

If M contains an arc PQR, such that P and R are on J, and all other points of the arc are either interior to J or exterior to J, then M contains three arcs, PQR, PSR, PTR, having only P and R in common. One arc, say PSR, lies within the domain bounded by the other two. Since $M-(PQR+PTR)$ is uniformly connected im kleinen relative to $(PQR+PTR)$, and since P is a limit point of points on the arc PSR interior to the simple closed curve $(PQR+PTR)$, there exists a positive number ε_1, which is less than the distance from P to any point of M exterior to the simple closed curve $(PQR+PTR)$. In the same way we can show the existence of positive numbers ε_2 and ε_3 which are less respectively than the distance from P to any point of M interior to $(PQR+PSR)$ and the distance from P to any point of M interior to $(PSR+PTR)$. Let us now select a positive number α which is less than the smallest of the numbers ε_1, ε_2, ε_3 and also less than the distance from P to R. Let us also select a point U, on the arc PSR, such that the diameter of the arc PU of PSR is less than $\alpha/4$. We shall now show that $M-PU$ is not uniformly connected im kleinen relative to PU. For, if we let $\varepsilon = \alpha/4$, no matter what value δ is given we can always select a point X, different from P on PQR, and a point Y, different from P on PTR, such that the distances from P to X and P to Y are less than $\delta/2$ and also less than $\alpha/4$. The distance from X to Y is less than δ, and yet any connected subset N of M containing X and Y either contains R or contains a point not on PQR, PSR, or PTR (in either case, N contains a point whose distance from P is more than α) and therefore N cannot lie within a circle of radius $\alpha/4$, nor is every point of N at a distance less than $\alpha/4$ from a point of PU. We have thus shown that the assumption that the arc PQR exists, leads to a contradiction.

If there are points of M not on J, and there are no arcs in M joining two points of J, save arcs that lie wholly in J, then any point Q of $M-J$ lies in a maximal connected subset T of $M-J$, which is closed save for a single limit point P on J. Let R be another point of J, and let us select a positive number α which is less than the distance from P to R. Let us also select a point U of T such that an arc PU exists in T of diameter less than $\alpha/4$. Then we shall show that $M-PU$ is not uniformly connected im kleinen relative to PU by a proof exactly the same as in the preceding paragraph, save that N in this case necessarily con-

tains R. We have thus shown that the assumption that there are points of M not on J leads to a contradiction. Therefore, in case I, $M = J$.

CASE II. M contains no simple closed curve.* Let AB be an arc in M. If AB contains a point P, $A \neq P \neq B$, which is a limit point of $M - AB$, and if T is any maximal connected subset of $M - AB$ having P as a limit point, then $M - (T + P)$ is disconnected, one maximal connected subset of $M - (T + P)$ containing $AP - P$, another containing $PB - P$. Since P is a common limit point of these two subsets of $M - (T + P)$, it is evident that $M - (T + P)$ is not uniformly connected im kleinen relative to $(T + P)$; for if we select X and Y according to our previous method, there is no connected subset of $M - (T + P)$ containing both X and Y. We have thus shown that the assumption that the point P exists leads to a contradiction. The points A and B may be limit points of $M - AB$ so that M consists of a set of arcs having only their end points in common. Therefore, in case II, M is a simple continuous arc, a ray, or an open curve, and the theorem is proved.

In conclusion, I wish to point out that the theorem is not true if *$M - W$ is uniformly connected im kleinen relative to W* is replaced by either the weaker condition, *$M - W$ is connected im kleinen*, or the independent condition *$M - W$ has the S-property*. For example, under these hypotheses M might consist of three arcs having only their end points in common.

* For a discussion of this type of continuous curve see S. Mazurkiewicz, loc. cit., pp. 119–130, and R. L. Wilder, Bull. Amer. Math. Soc., vol. 29 (1923), p. 118.

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